5.1 An Improved Estimator in Systematic Sampling

5.1.1 Introduction and Review Literature

There are some natural populations like forest etc., where it is not possible to apply easily the simple random sampling or other sampling schemes for estimating the population characteristics. In such situations, one can easily implement the method of systematic sampling for selecting a sample from the population. Systematic sampling has the advantage of selecting the whole sample with just a random start. Estimation in systematic sampling has been discussed in detail by Lahiri (1954), Gautschi (1957), Hajeck (1959) and Cochran (1957). Use of auxiliary information in construction of estimators is considered by Kushwaha and Singh (1989), Banarasi et al. (1993) and Singh and Singh (1998). Further we introduced the following terminology to discuss the properties of estimators:

Let $y$ be the study variable and $x$ be the auxiliary variable defined on a finite population $U = (U_1, U_2, ..........., U_N)$. Here, we assume $N = nk$, where $n$ and $k$ are positive integers. Let $(y_{ij}, x_{ij}; i=1,2,...,k; j=1,2,...,n)$ denote the value of $j^{th}$ unit in the $i^{th}$ sample. The systematic sample means

$$\bar{y}^* = \frac{1}{n} \sum_{j=1}^{n} y_{ij}, \quad \bar{x}^* = \frac{1}{n} \sum_{j=1}^{n} x_{ij}$$

are unbiased estimators of the population means $(\overline{y}, \overline{x})$ of $(y, x)$, respectively.

If $e_0 = \left( \frac{\bar{y}^* - \overline{y}}{\overline{y}} \right)$ and $e_1 = \left( \frac{\bar{x}^* - \overline{x}}{\overline{x}} \right)$

Then, we have

$E(e_0) = E(e_1) = 0,$

and
\[
E(\epsilon_0^2) = \theta (1 + (n - 1)\rho_y) C_y^2, \quad E(\epsilon_1^2) = \theta (1 + (n - 1)\rho_x) C_x^2
\]
\[
E(\epsilon_0\epsilon_1) = \theta (1 + (n - 1)\rho_y) \frac{1}{2} (1 + (n - 1)\rho_x) \frac{1}{2} \rho C_y C_x,
\]
where, \( \theta = \frac{N - 1}{N n} \),
\[
\rho_x = \frac{E(x_{ij} - \bar{X})(x_{ij} - \bar{X})}{E(x_{ij} - \bar{X})^2}, \quad \rho_y = \frac{E(y_{ij} - \bar{Y})(y_{ij} - \bar{Y})}{E(y_{ij} - \bar{Y})^2}
\]
and \( \rho = \frac{E(x_{ij} - \bar{X})(y_{ij} - \bar{Y})}{E(x_{ij} - \bar{X})^2(y_{ij} - \bar{Y})^2} \).

\((C_y, C_x)\) are the coefficients of variation of the variates \((y, x)\) respectively. It is assumed that the population mean \( \bar{X} \) of the auxiliary variable is known. The classical ratio and product estimators for \( \bar{Y} \) based on the systematic sample \((y_{ij}, x_{ij}; i = 1, 2, ..., k; j = 1, 2, ..., n)\) of size \( n \), are, respectively, defined by

\[
t_1 = \left( \frac{\bar{y}^*}{\bar{x}} \right) \bar{X} \tag{5.1}
\]
\[
t_2 = \bar{y}^* \left( \frac{\bar{x}}{\bar{X}} \right) \tag{5.2}
\]
which are respectively due to Swain (1964) and Shukla (1971).

The variances of \( t_1 \) and \( t_2 \) to the first order of approximation are, respectively, given by

\[
\text{Var}(t_1) = \theta \bar{Y}^2 (1 + (n - 1)\rho_x) \left( \rho^* C_y^2 + (1 - 2k\rho^*) C_x^2 \right) \tag{5.3}
\]
\[
\text{Var}(t_2) = \theta \bar{Y}^2 (1 + (n - 1)\rho_x) \rho^* C_y^2 + (1 + 2k\rho^*) C_x^2 \tag{5.4}
\]

And the variances of usual unbiased estimator \( \bar{y}^* \) is given by

\[
\text{Var}\left( \bar{y}^* \right) = \theta \bar{Y}^2 (1 + (n - 1)\rho_y) C_y^2 \tag{5.5}
\]
where \( \rho^* = \frac{1 + (n - 1)\rho_y}{1 + (n - 1)\rho_x}, \quad k = \frac{c_y}{c_x}. \)
5.1.2 Adapted Estimator

Bahl and Tuteja (1991) introduced an exponential ratio type estimator for population mean of \( y \) in simple random sampling given by
\[
t_{3\text{exp}} = \bar{y} \exp \left( \frac{X - \bar{x}}{X + \bar{x}} \right)
\]  
(5.6)

Motivated by Bahl and Tuteja (1991), we adapt this estimator to the systematic sampling as
\[
t_3 = \bar{y}^* \exp \left( \frac{X - \bar{x}^*}{X + \bar{x}^*} \right)
\]  
(5.7)

Expressing equation (5.7) in terms of \( e \)’s, we have
\[
t_3 = \bar{Y}(1 + e_0) \exp \left[ -\frac{e_i}{2} \left( 1 + \frac{e_i}{2} \right)^{-1} \right]
\]  
(5.8)

\[
t_3 = \bar{Y} \left[ 1 + e_0 - \frac{e_i}{2} + \frac{3e_i^2}{8} - \frac{e_0 e_i}{2} \right]
\]  
(5.9)

Subtracting \( \bar{Y} \) from both the sides of equation (5.9) and then taking expectation of both sides, we get the bias of the estimator \( t_3 \), up to the first order of approximation, as
\[
t_3 - \bar{Y} = \bar{Y} \left[ e_0 - \frac{e_i}{2} + \frac{3e_i^2}{8} - \frac{e_0 e_i}{2} \right]
\]  
(5.10)

Taking expectation on both sides in (5.10) we get the bias of the estimator \( t_3 \) as
\[
B(t_3) = \bar{Y} \theta \left[ \frac{3C_x^2}{8} - \frac{C_0 C_1}{2} \right]
\]  
(5.11)

\[
C_0^2 = \left( 1 + (n - 1) \rho_y \right) C_y^2
\]

Where, \( C_1^2 = \left( 1 + (n - 1) \rho_x \right) C_x^2 \) 

\[
C_0 C_1 = \left( 1 + (n - 1) \rho_y \right)^{1/2} \left( 1 + (n - 1) \rho_x \right)^{1/2} \rho C_y C_x
\]  
(5.12)

Squaring both sides of (5.10) and then taking expectation, we get the MSE of the estimator up to the first order of approximation, as
\[
\text{MSE}(t_3) = \bar{Y} \theta \left[ c_0^2 + \frac{c_1^2}{4} - C_0 C_1 \right]
\]  
(5.13)

Following Bahl and Tuteja (1991), using a transformation proposed by Bedi (1996), Shabbir and Gupta (2011) introduced an exponential ratio type estimator for population mean in simple random sampling given by,
\[ t_4 = \left[ k_1 y + k_2 (\bar{X} - \bar{x}) \right] \exp \left( \frac{(\bar{A} - \bar{a})}{(A + \bar{a})} \right) \]  

(5.14) 

where, \( k_i \) (i = 1, 2) are constants. 

Let \( a_i = x_i + N\bar{X} \). 

So we have \( \bar{a} = \bar{x} + N\bar{X} \) and \( \bar{A} = \bar{X} + N\bar{X} \) 

Motivated by Shabbir and Gupta (2011), we adapt this family to the systematic sampling as 

\[ t_4 = \left[ k_1 y^* + k_2 (\bar{X} - \bar{x}^*) \right] \exp \left( \frac{(A - \bar{a}^*)}{(A + \bar{a}^*)} \right) \]  

(5.15) 

where \( \bar{a}^* = \bar{x}^* + N\bar{X} \). 

Expressing \( t_4 \) in terms of \( e_i \)'s, we have 

\[ t_4 = [k_1 y(1 + e_0) - k_2 \bar{X} e_1] \exp \left( -\frac{e_1}{2(1 + N)} \left[ 1 + \frac{e_1}{2(1 + N)} \right]^{-1} \right) \]  

(5.16) 

Expanding the right hand in (5.16), to the 1st order of approximation, we obtain 

\[ t_4 - \bar{Y} \approx \bar{Y} \left[ (k_1 - 1) + k_1 \left( e_0 \frac{e_1}{2(1 + N)} - \frac{e_0 e_1}{2(1 + N)} + \frac{3e_1^2}{8(1 + N)^2} \right) \right] - k_2 \bar{X} e_1 \frac{e_1^2}{2(1 + N)^2} \]  

(5.17) 

Taking expectation of both sides in (5.17), we get the bias of the estimator \( t_4 \) as 

\[ B(t_4) = \bar{Y} \left[ (k_1 - 1) + k_1 \left( \frac{3C^2}{8(1 + N)^2} - \frac{C_0C_1}{2(1 + N)} \right) \right] - k_2 \bar{X} e_1 \frac{C^2}{2(1 + N)^2} \]  

(5.18) 

Squaring (5.17) and then taking expectation, we get the MSE of the estimator \( t_4 \) 

\[ \text{MSE}(t_4) = \bar{Y}^2 \left[ (k_1 - 1)^2 + k_1^2 \left( \frac{C_0^2}{(1 + N)^2} + \frac{C_1^2}{(1 + N)^2} - 2k_1 \left( \frac{3}{8(1 + N)^2} - \frac{C_0C_1}{2(1 + N)^2} \right) \right) \right] 
\]  

\[ + k_2^2 \bar{X}^2 C_1^2 \theta - \frac{k_2 \bar{X} Y C_1^2 \theta}{(1 + N)} + 2k_2 \bar{X} C_{1/2} \]  

\[ \left[ \bar{Y} \left( \frac{C_1^2}{(1 + N)} - C_0 C_1 \right) \right] \]  

(5.19) 

Where, \( k_1 = \frac{A_1 A_2 - \bar{Y}^2 (1 + A_2) A_3}{2A_2^2 - 2A_3 \bar{Y}^2 (1 + A_1)} \), \( k_2 = \frac{\bar{Y}^2 \left( 2A_4 (1 + A_2) - A_4 (1 + A_1) \right)}{2A_4^2 - 2A_3 \bar{Y}^2 (1 + A_1)} \) 

and \( A_1 = \theta \left( \frac{C_0^2}{(1 + N)^2} + \frac{C_1^2}{(1 + N)^2} - 2C_0 C_1 \right) \), \( A_2 = \theta \left( \frac{3C_0^2}{8(1 + N)^2} - \frac{C_0 C_1}{2(1 + N)^2} \right) \), 
\[ A_3 = \bar{X}^2 \theta C_1^2 \), \( A_4 = \bar{Y} \bar{X} \theta \left( \frac{C_1^2}{(1 + N)^2} - C_0 C_1 \right) \), \( A_5 = \bar{Y} \bar{X} \theta \left( \frac{3C_0^2}{8(1 + N)^2} - \frac{C_0 C_1}{2(1 + N)^2} \right) \).
5.1.3 Empirical Study

In the support of theoretical results, we have considered the data given in Murthy (1967, p. 131-132). These data are related to the length and timber volume for ten blocks of the black’s mountain experimental forest. The value of intraclass correlation coefficients $\rho_x$ and $\rho_y$ have been given approximately equal by Murthy (1967, p. 149) and Kushwaha and Singh (1989) for the systematic sample of size 16 by enumerating all possible systematic samples after arranging the data in ascending order of strip length. The particulars of the population are given below:

\[
N = 176, \quad n = 16, \quad \overline{Y} = 282.6136, \quad \overline{X} = 6.9943, \\
S^2_Y = 24114.6700, \quad S^2_X = 8.7600, \quad \rho = 0.8710.
\]

Table 5.1 PRE of estimators with respect to $\overline{y}$

<table>
<thead>
<tr>
<th>Estimators</th>
<th>PRE ($\overline{y}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{y}$</td>
<td>100.00</td>
</tr>
<tr>
<td>$t_1$</td>
<td>397.5</td>
</tr>
<tr>
<td>$t_2$</td>
<td>34.07</td>
</tr>
<tr>
<td>$t_3$</td>
<td>209.43</td>
</tr>
<tr>
<td>$t_4$</td>
<td>468.68</td>
</tr>
</tbody>
</table>

5.1.4 Conclusion

In section 5.1.2, we have adapted Bahl and Tuteja (1991) estimator in Systematic sampling. In support of theoretical findings, a numerical study is also carried out to demonstrate the superiority of suggested method over existing one. Further, it is observed from the Table 5.1 that the proposed estimator $t_4$ under optimum condition performs better than the Swain (1964) estimator $t_1$ and Shukla (1971) estimator $t_2$ and $t_3$. 
5.2 Some Improved Suggestions in Circular Systematic Sampling under fully and Partially Respondent Case

In sample surveys, auxiliary information on the finite population under study is quite often available from previous experience, census or administrative databases. The sampling literature describes a wide variety of techniques for using auxiliary information to improve the sampling design and/or obtain more efficient estimators. It is well known that when the auxiliary information is to be used at the estimation stage, the ratio, product and regression methods are widely employed.

Systematic sampling technique is operationally more convenient than simple random sampling. It also ensure at the same time each unit has equal probability of inclusion in the sample. In this method of sampling, the first unit is selected with the help of random number and remaining units are selected automatically according to predetermined pattern. Hasel (1942) and Griffith (1945-1946) found systematic sampling to be efficient and convenient in sampling certain natural populations like forest areas for estimating the volume of the timber and area under different types of cover. Cochran (1946) and Hajeck (1959) had stated that in large-scale sampling work, this procedure provides more efficient estimators than those provided by simple random sampling and/or stratified random sampling for certain populations.

When the population mean $\bar{X}$ of the auxiliary variable $x$ is known, Swain (1964) and Shukla (1971) have suggested the ratio and product estimators for the population mean $\bar{Y}$ of the survey variable $y$, respectively, along with their properties in systematic sampling. Some other notable work in this area are Singh and Singh (1998), Singh R. et al. (2012), Singh and Jatwa (2012), Singh and Solanki (2012) and Verma et al. (2014).

In linear systematic sampling, given a sample size $n$, sampling is possible only if population size $N$ is divisible by $n$. Even when this condition is satisfied, the scheme cannot provide estimate of variance of the sample mean. This scheme has two drawbacks namely, given $N$, $n$ has limited choice and variance of the sample mean is not estimable. The first limitation could be removed through circular systematic sampling as suggested by Lahiri (1952). The procedure consists in selecting a unit, by a random start, from 1 to $N$ and then thereafter selecting every $k^{th}$ unit, $k$ being an integer nearest to $N/n$, in a circular
manner, until a sample of n units is obtained. Suppose that a unit with random number \( i \) is selected. The sample will then consists of the units corresponding to the serial numbers

\[
\text{Label}=\begin{cases} 
  i + jk, & \text{for } i = 0,1,\ldots,(n-1), \\
  i + jk - N, & \text{for } N < i + jk.
\end{cases}
\]

[For details see Singh and Chaudhary (1986, pp 83)].

In the following manner, we may draw \( N \) circular systematic samples, each of size \( n \) as displayed in table 5.2

<table>
<thead>
<tr>
<th>Sample No.</th>
<th>1</th>
<th>2</th>
<th>\ldots</th>
<th>i</th>
<th>\ldots</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( u_1 )</td>
<td>( u_2 )</td>
<td>( \ldots )</td>
<td>( u_i )</td>
<td>( \ldots )</td>
<td>( u_N )</td>
</tr>
<tr>
<td></td>
<td>( u_{k+1} )</td>
<td>( u_{k+2} )</td>
<td>( \ldots )</td>
<td>( u_{k+i} )</td>
<td>( \ldots )</td>
<td>( u_{k+N} )</td>
</tr>
<tr>
<td></td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td></td>
<td>( u_{(n-1)k+1} )</td>
<td>( u_{(n-1)k+2} )</td>
<td>( \ldots )</td>
<td>( u_{(n-1)k+i} )</td>
<td>( \ldots )</td>
<td>( u_{(n-1)k+N} )</td>
</tr>
</tbody>
</table>

From this \( N \) possible sample, a sample of size \( n \) is selected randomly to observe \( Y \) and \( X \).

The present problem aims to gives some contribution on this subject. For this purpose, taking motivation from Sahai and Ray (1980), Koyunsu and Kadilar (2010), Singh and Solanki (2013) and Singh and Malik (2014), three modified classes of estimators are suggested for the population mean \( \bar{Y} \) using auxiliary information in circular systematic sampling design, following this we have also studied the effect of non-response (present in study variable \( y \)) on suggested estimators.

### 5.2.1 Processing and Notations Used In Circular Systematic Sampling

Let us suppose that \( U^* \) be a finite population consists of \( N \) distinct labeled units i.e. \( U^* = (U_1, U_2, \ldots, U_N) \) and \( n \) be a fixed sample size.

Also, let \( Y \) and \( X \) be study and auxiliary variables taking values \( y_{ij} \) and \( x_{ij} \) \( i = (1,2,\ldots, N), j = (1,2,\ldots, n) \).
The CSS sample means $\bar{y}_{CSS} = \frac{\sum_{j=1}^{n} y_{ij}}{n}$ and $\bar{x}_{CSS} = \frac{\sum_{j=1}^{n} x_{ij}}{n}$ are unbiased estimates of population means $\bar{Y} = \frac{\sum_{j=1}^{N} y_{ij}}{N}$ and $\bar{X} = \frac{\sum_{j=1}^{N} x_{ij}}{N}$ respectively.

The variances of $y_{CSS}$ and $x_{CSS}$ under CSS design are written respectively as:

$$V(\bar{y}_{CSS}) = \theta_1 \left[ 1 + (n-1)\rho_y \right] \frac{S_y^2}{n} = \frac{\tilde{S}_y^2}{n} = \bar{Y}^2 \tilde{C}_y^2$$

and

$$V(\bar{x}_{CSS}) = \theta_1 \left[ 1 + (n-1)\rho_x \right] \frac{S_x^2}{n} = \frac{\tilde{S}_x^2}{n} = \bar{X}^2 \tilde{C}_x^2$$

where

$$\theta_1 = \left( \frac{N-1}{N} \right), \quad S_y^2 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \bar{Y})^2}{n(N-1)}, \quad S_x^2 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (x_{ij} - \bar{X})^2}{n(N-1)}$$

With

$$\rho_y = \frac{2}{n(n-1)(N-1)} S_y^2 \sum_{i=1}^{N} (y_{ij} - \bar{Y})(y_{iu} - \bar{Y})$$

and

$$\rho_x = \frac{2}{n(n-1)(N-1)} S_x^2 \sum_{i=1}^{N} (x_{ij} - \bar{X})(x_{iu} - \bar{X})$$

where $(\rho_y, \rho_x)$ represents intraclass correlation coefficients between pairs of units within the CSS for variable $Y$ and $X$, respectively.

Also,

$$\text{Cov}(\bar{y}_{CSS}, \bar{x}_{CSS}) = \theta_1 \left[ 1 + (n-1)\rho_y \right]^{1/2} \left[ 1 + (n-1)\rho_x \right]^{1/2} \frac{S_{yx}}{n} = \frac{\tilde{S}_{yx}}{n} = \bar{Y} \bar{X} \tilde{C}_{yx}$$

where

$$S_{yx} = \frac{\sum_{i=1}^{N} \sum_{j=1}^{n} (y_{ij} - \bar{Y})(x_{ij} - \bar{X})}{n(N-1)}.$$
5.2.2 Estimators in Literature

It is well known that, regression estimators are always more efficient than usual mean and ratio estimators, at least asymptotically, thus we consider linear regression estimators based on CSS as standard result for making comparison with our suggested class of estimators.

The linear regression estimator of the population mean $\bar{Y}$ based on CSS with known $\bar{X}$ is defined as

$$\bar{y}_{1r}(c) = \bar{y}_{CSS} + \hat{\beta}_{yx}(\bar{X} - \bar{x}_{CSS}),$$

(5.20)

where $\hat{\beta}_{yx} = \frac{s_{yx}}{s_x^2}$ is an estimator for population regression coefficient $\beta_{yx}$ with

$$s_x^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x}_{CSS})^2}{(n-1)}$$

and

$$s_{yx} = \frac{\sum_{j=1}^{n} (y_{ij} - \bar{y}_{CSS})(x_{ij} - \bar{x}_{CSS})}{(n-1)}$$

The asymptotic variance of $\bar{y}_{u}(c)$ is respectively given by

$$AV[\bar{y}_{u}(c)] = S_y^2 (1 - \rho_{yx}^2),$$

(5.21)

where, $\hat{\rho}_{yx}^2 = \frac{S_{yx}^2}{S_y S_x}$

When population mean $\bar{X}$ is unknown, double sampling scheme is used. Under double sampling scheme, first we divide the population into $N$ clusters of size $n$, each according to CSS, and select randomly $m$ distinct clusters ($1 < m < k$) to estimate $\bar{X}$ only. In second phase, a cluster is selected randomly from $m$ CSSs to estimate $\bar{Y}$. Hence, the expression for $\bar{y}_{1r}(c)$, with unknown $\bar{X}$, is given as

$$\bar{y}_{1r}(c) = \bar{y}_{CSS} + \hat{\beta}_{yx}(\bar{x}'_{CSS} - \bar{x}_{CSS}),$$

(5.22)

where

$$\bar{x}'_{CSS} = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$$

It is known that,

$$V(\bar{x}') = \text{Cov}(\bar{x}'_{CSS}, \bar{x}_{CSS}) = \theta_1 \left[ l + (n-1)\rho_x \right] S_x^2 = \frac{S_x^2}{nm} = \frac{S_x^2}{m}$$

and
The asymptotic variance of $\bar{y}'(c)$ is given by

$$ AV(\bar{y}'(c)) = \tilde{S}_y^2 \left( 1 - \tilde{\rho}_{yx}^2 \right) $$

(5.23)

where, $\tilde{\rho}_{yx}^2 = \frac{\tilde{S}_{yx}^2}{\tilde{S}_y^2 \tilde{S}_x^2}$, such that

$$ \tilde{S}_{yx} = \left( \frac{m-1}{m} \right) \tilde{S}_{yx} \quad \text{and} \quad \tilde{S}_x^2 = \left( \frac{m-1}{m} \right) \tilde{S}_x^2 $$

(5.24)

with

$$ \tilde{C}_x = \frac{\tilde{S}_x^2}{\bar{Y}^2} \quad \text{and} \quad \tilde{C}_{yx} = \frac{\tilde{S}_{yx}}{\bar{Y} \bar{X}} $$

### 5.2.3 Suggested Family of Estimators

Much literature has been written on sampling from finite populations to address the issue of the efficient estimation of the mean (or total) of a survey variable when auxiliary variables are available. Our analysis refers to simple random sampling without replacement (SRSWOR) and circular systematic sampling (CSS) considers, for brevity, the case when only a single auxiliary variable is used.

In this section, we have suggested three modified class of estimators for estimating $\bar{Y}$ using prior knowledge on $x$, based on SRSWOR and circular systematic sampling. The suggested modified classes are motivated from Singh and Solanki (2013), Singh and Malik (2014), Sahai and Ray (1980) and Koyuncu and Kadilar (2010). After observing theoretical and empirical results, we infer that the suggested estimators performs better than usual linear regression estimators in both SRSWOR and CSS design, which demonstrate the superiority of suggested estimators. Also, we observed that CSS design is more convenient and efficient than SROWOR design.

#### 5.2.3 (a) Difference-Type Class of Estimator

Motivated by Singh and Solanki (2013) and Koyuncu and Kadilar (2010), we propose the following difference-type class of estimators for estimating population mean $\bar{Y}$ of a survey variable under CSS assuming $\bar{X}$ is known
\[ T_1 = \left[ \kappa_1 \bar{y}_{\text{CSS}} + \kappa_2 \bar{x}_{\text{CSS}} + (l^{-1} \kappa_1^{-1} \kappa_2) \bar{X} \right] \exp \left[ \alpha \frac{\bar{X}^* - \bar{x}_{\text{CSS}}^*}{\bar{X}^* + \bar{x}_{\text{CSS}}^*} \right] \] (5.25)

where \((\kappa_1, \kappa_2)\) are suitably chosen scalars to be properly determined for minimum mean square error (MSE) of suggested estimators, \(\bar{x}_{\text{CSS}}^* = \eta \bar{x}_{\text{CSS}} + \lambda, \bar{X}^* = \eta \bar{X} + \lambda\) with \((\eta, \lambda)\) are either constants or function of some known population parameters and \(\alpha\) being constants which take finite values for designing the different estimators (see Singh and Kumar (2011)). To obtain the bias and \(MSE\) expressions of the proposed class of estimators \(T_1\), we define

\[ \delta_y = \frac{(\bar{y}_{\text{CSS}} - \bar{Y})}{\bar{Y}} \quad \text{and} \quad \delta_x = \frac{(\bar{x}_{\text{CSS}} - \bar{X})}{\bar{X}} \]

Such that

\[
\begin{align*}
E(\delta_y) &= E(\delta_x) = 0 \\
E(\delta_y^2) &= \tilde{C}_y^2, E(\delta_x^2) &= \tilde{C}_x^2 \quad \text{and} \quad E(\delta_y \delta_x) = \tilde{C}_{yx} 
\end{align*}
\] (5.26)

The expressions for asymptotic bias and asymptotic variance of the suggested estimator \(T_1\) using CSS are given respectively as

\[
\begin{align*}
\text{AB}(T_1) &= \left[ \phi + \phi_1 \tilde{C}_x^2 + \kappa_1 \left( A_2 \phi_2 - A - \frac{\bar{Y} \alpha \tau_1 \tilde{C}_{yx}}{2} \right) - \kappa_2 \eta \bar{X} \alpha \tau_1 \tilde{C}_x^2 \right] \\
\text{AV}(T_1) &= \left[ N + 1 \kappa_1^2 N_{11} + 1 \kappa_2^2 N_{12} - 2 \kappa_1 N_{13} - 2 \kappa_2 N_{14} + 2 \kappa_1 \kappa_2 N_{15} \right] 
\end{align*}
\] (5.27, 5.28)

Minimisation of \(\text{AV}(T_1)\) is achieved for the optimum choice of constants \(\kappa_1\) and \(\kappa_2\)

\[
\kappa_1^\text{(opt)} = \kappa_1^* = \left( \frac{N_{12} N_{13} - N_{14} N_{15}}{N_{11} N_{12} - N_{15}^2} \right) \quad \text{and} \quad \kappa_2^\text{(opt)} = \kappa_2^* = \left( \frac{N_{11} N_{14} - N_{13} N_{15}}{N_{11} N_{12} - N_{15}^2} \right)
\]

where

\[
\begin{align*}
N &= \phi^2 + \frac{\alpha^2 \tau_1^2 \bar{X}^2 \tilde{C}_x^2}{2} + 2 \phi \phi_1 \tilde{C}_x^2, N_{11} = \bar{Y}^2 \tilde{C}_y^2 + \phi^2 + \frac{\alpha^2 \tau_1^2 \phi^2}{4} \tilde{C}_x^2 - 2 \phi \phi_1 \tilde{C}_x^2 + 2 \alpha \tau_1 \phi \bar{Y} \tilde{C}_{yx} \\
N_{12} &= \eta^2 \bar{X}^2 \tilde{C}_x^2, N_{13} = \phi^2 + \frac{\alpha \tau_1 \bar{Y} (\phi + \bar{X}^*)}{2} \tilde{C}_{yx} + \frac{\alpha^2 \tau_1^2 \bar{X}^* \phi}{2} \tilde{C}_x^2 + \phi \tilde{C}_x^2 (\phi_1 - \phi_2) \\
N_{14} &= \eta \bar{X} \phi \bar{X} \tilde{C}_x^2 / 2, N_{15} = \eta \bar{X} \phi \bar{X} \tilde{C}_x^2 + \eta \bar{X} \bar{Y} \tilde{C}_{yx} 
\end{align*}
\]

Such that, \(\tau_1 = \frac{\eta \bar{X}}{\eta \bar{X} + \lambda}, \phi = (\bar{X}^* - \bar{Y}), \phi_1 = \left( \frac{\alpha \tau_1 \bar{X}^*}{4} + \frac{\alpha^2 \tau_1^2 \bar{X}^*}{8} \right)\)

and \(\phi_2 = \left( \frac{\alpha \tau_1^2 \bar{Y}}{4} + \frac{\alpha^2 \tau_1 \bar{Y}}{8} - \frac{\alpha^2 \tau_1^2 \bar{X}^*}{4} - \frac{\alpha^2 \tau_1^2 \bar{X}^*}{8} \right)\).
Now suppose $\bar{x}$ is unknown, the analogue of $T_i$ becomes

$$T'_i = \left[ \kappa_{d1}\bar{x}_{CSS} + \kappa_{d2}\bar{x}'_{CSS} + (1-\kappa_{d1}-\kappa_{d2})\bar{x}'_{CSS} \right] \exp \left[ \alpha \frac{\bar{x}'_{CSS} - \bar{x}_{CSS}}{\bar{x}'_{CSS} + \bar{x}_{CSS}} \right]$$

(5.29)

where, the notations used here are already defined earlier.

To obtain the bias and $MSE$ of the proposed class of estimators $T_i$, we define

$$\delta'_x = \frac{(\bar{x}'_{CSS} - \bar{x})}{\bar{x}}$$

Such that

$$E(\delta'_x) = 0, \quad E(\delta'_x^2) = E(\delta'_x, \delta'_x) = \frac{\tilde{C}^2}{m} \quad \text{and} \quad E(\delta'_y, \delta'_x) = \frac{\tilde{C}_{yx}}{m}$$

(5.30)

The expressions for asymptotic bias and asymptotic variance of the suggested estimator $T'_i$ using CSS are given respectively as

$$AB(T'_i) = \left[ \phi + \phi_1 \tilde{C}_x + \kappa_{d1} \left( A_2\phi_2 - \frac{\bar{X}\alpha\tau_1\tilde{C}_x}{2} \right) - \frac{\eta_x\alpha\tau_2\tilde{C}_x}{2} \right]$$

and

$$AV(T'_i) = \left[ N + \left( \kappa_{d1}^2 N_{1,1} + \kappa_{d2}^2 N_{1,2} - 2\kappa_{d1} N_{1,3} - 2\kappa_{d2} N_{1,4} + 2\kappa_{d1} \kappa_{d2} N_{1,5} \right) \right]$$

(5.32)

Minimisation of $AV(T'_i)$ is achieved for the optimum choice of constants $\kappa_{d1}$ and $\kappa_{d2}$

$$\kappa_{d1}^* = \frac{N_{1,2} N_{1,3} - N_{1,4} N_{1,5}}{N_{1,1} N_{1,2} - N_{1,3}^2} \quad \text{and} \quad \kappa_{d2}^* = \frac{N_{1,1} N_{1,4} - N_{1,3} N_{1,5}}{N_{1,1} N_{1,2} - N_{1,3}^2}$$

where,

$$N = \phi^2 + \frac{\alpha^2\tau_1^2\bar{X}'^2\tilde{C}_x^2}{2} + 2\phi\phi_1 \tilde{C}_x^2, \quad N_{1,1} = \bar{Y}\tilde{C}_y^2 + \phi^2 + \frac{\alpha^2\tau_1^2\phi^2}{4} \tilde{C}_x^2 - 2\phi\phi_2 \tilde{C}_x^2 + 2\alpha\tau_1\phi\bar{Y}\tilde{C}_{yx}$$

$$N_{1,2} = \eta^2\bar{X}'^2\tilde{C}_x^2, \quad N_{1,3} = \phi^2 + \frac{\alpha\tau_1 \bar{Y}(\phi + \bar{X}')}{2} \tilde{C}_{yx} + \frac{\alpha^2\tau_1^2\bar{X}'^2\phi}{2} \tilde{C}_x^2 + \phi \tilde{C}_x^2 (\phi_1 - \phi_2)$$

$$N_{1,4} = \eta\alpha\tau_1 \bar{X}(\phi + \bar{X}') \tilde{C}_x^2 / 2, \quad N_{1,5} = \eta\alpha\tau_1 \phi \bar{X}\tilde{C}_x^2 + \eta\bar{X}\bar{Y}\tilde{C}_{yx}$$

with, $\tilde{C}_x^2$ and $\tilde{C}_{yx}$ which is already defined in equation (5.24).

**Note 1:** It can be observed from equation (5.27), (5.28) and (5.31), (5.32) that the asymptotic bias and minimum asymptotic variance of $T_i$ and $T'_i$ look similar. However, due to single and double sampling design the dissimilarity exists only in terms $\left( \tilde{C}_x^2, \tilde{C}_{yx} \right)$ and $\left( \tilde{C}_x^2, \tilde{C}_{yx} \right)$. 80
5.2.3 (b) Difference-cum-Exponential Type Class of Estimators

Motivated by Sahai and Ray (1980) and Singh and Malik (2014), the following modified class of estimators has been suggested for the population mean $\bar{Y}$ assuming that $\bar{X}$ is known

$$T_2 = \bar{Y}_{CSS} \left[ 2 \kappa_1 + 2 \kappa_2 \left( 2 - \frac{\bar{X}_{CSS}}{\bar{X}} \right) \right] \exp \left[ \gamma \frac{(\mu \bar{X} + \nu) - (\mu \bar{X}_{CSS} + \nu)}{(\mu \bar{X} + \nu) + (\mu \bar{X}_{CSS} + \nu)} \right]$$  (5.33)

where $\beta$, $\mu$ and $\nu$ are either real numbers or function of known parameters of auxiliary variable $x$ such as $S_x$, $C_x$, $\rho_x$ etc. The scalars $\gamma$ takes values -1,0 and -1 for ratio and product type estimators, respectively.

The expressions for asymptotic bias and asymptotic variance of the suggested estimator $T_2$ using CSS are given as

$$AB(T_2) = \bar{Y} \left[ 2 \kappa_1 \left( 1 + \phi_3 \bar{C}_x^2 - \frac{\gamma \tau_2 \bar{C}_{yx}}{2} \right) \right] + 2 \kappa_2 \left( 1 + \phi_3 \bar{C}_x^2 - \phi_5 \bar{C}_{yx} \right) - 1$$  (5.34)

and

$$AV(T_2) = \bar{Y} \left[ 2 \kappa_1 N_{21} + 2 \kappa_2 N_{22} - 2 \kappa_1 N_{23} - 2 \kappa_2 N_{24} + 2 \kappa_1^2 \kappa_2 N_{25} \right]$$  (5.35)

Minimisation of $AV(T_2)$ is achieved for the optimum choice of constants $\kappa_1$ and $\kappa_2$

$$\kappa_1^* = \left( \frac{N_{22} N_{23} - N_{24} N_{25}}{N_{21} N_{22} - N_{25}^2} \right) \quad \text{and} \quad \kappa_2^* = \left( \frac{N_{21} N_{24} - N_{23} N_{25}}{N_{21} N_{22} - N_{25}^2} \right)$$

where

$$N_{21} = \left[ 1 + \bar{C}_y^2 + \left( \frac{\gamma^2 \tau_2^2}{4} + 2 \phi_3 \right) \bar{C}_x^2 - 2 \gamma \tau_2 \bar{C}_{yx} \right],$$

$$N_{22} = \left[ 1 + \bar{C}_y^2 + \left( \phi_5 + 2 \phi_4 \right) \bar{C}_x^2 - 4 \phi_5 \bar{C}_{yx} \right],$$

$$N_{23} = \left[ 1 + \phi_3 \bar{C}_x^2 - \frac{\gamma \tau_2 \bar{C}_{yx}}{2} \right], \quad N_{24} = \left[ 1 + \phi_4 \bar{C}_x^2 - \phi_5 \bar{C}_{yx} \right]$$

$$N_{25} = \left[ 1 + \bar{C}_y^2 + \phi_4 \bar{C}_x^2 - 2 \phi_3 \bar{C}_{yx} - \beta \tau_2 \bar{C}_{yx} + \frac{\gamma \tau_2 \phi_5 \bar{C}_x^2}{2} + \phi_3 \bar{C}_x^2 \right]$$

Such that,

$$\tau_2 = \frac{\mu \bar{X}}{\mu \bar{X} + \nu}, \quad \phi_3 = \left( \frac{\gamma \tau_2^2}{4} + \frac{\gamma^2 \tau_2^2}{8} \right), \quad \phi_4 = \left( \frac{\gamma \tau_2^2}{4} + \frac{\gamma^2 \tau_2^2}{8} + \frac{\beta \tau_2 \gamma}{2} - \frac{\beta (\beta - 1)}{2} \right)$$

81
and $\phi_2 = \frac{\gamma \tau_2}{2} + \beta$.

Now assuming that $\overline{X}$ is unknown, then the analogue of $T_2$ becomes

$$T_2' = \bar{y}_{CSS} \left[ 2 \kappa_{d1} + 2 \kappa_{d2} \left\{ 2 - \frac{\bar{X}_{CSS}}{\bar{X}} \right\} \beta \right] \exp \left[ \gamma \left( \frac{\mu \bar{X}_{CSS} + \nu}{(\mu \bar{X}_{CSS} + \nu) + (\mu \bar{X} + \nu)} \right) \right]$$  \hspace{1cm} (5.36)

The asymptotic bias and variance of $T_2'$ will be almost similar to $T_2$. The difference between the AB and AV of $T_2'$ and $T_2$ will be same like the difference between $T_1'$ and $T_1$ explained in earlier part of section 5.2.3. Therefore, replacing the terms $(\tilde{C}_x^2, \tilde{C}_{yx})$ in (5.34) and (5.35) by $(\tilde{C}_x^2, \tilde{C}_{yx})$, we get the asymptotic bias and minimum asymptotic variance of $T_2'$.

5.2.3 (c) Generalized Exponential-Ratio type estimators

Motivated by Sahai and Ray (1980) and Koyuncu and Kadilar (2010), the following modified class of estimators has been defined for the population mean $\overline{Y}$ assuming that $\overline{X}$ is known

$$T_3 = \bar{y}_{CSS} \left[ 3 \kappa_1 \left\{ 2 - \frac{\bar{X}_{CSS}}{\bar{X}} \right\} + 3 \kappa_2 \exp \left\{ \frac{\eta_1 (\bar{X} - \bar{X}_{CSS})}{(\bar{X} + \bar{X}_{CSS}) + 2 \lambda_1} \right\} \right]$$  \hspace{1cm} (5.37)

where $(3 \kappa_1, \kappa_2, \beta_1)$ are defined earlier and $(\eta_1, \lambda_1)$ being constants, which take finite real values or fraction of known parameters for designing the different family of proposed estimators.

The expressions for asymptotic bias and asymptotic variance of the suggested estimator $T_3$ using CSS are given as

$$AB(T_3) = \bar{y} \left[ 3 \kappa_1 \left\{ 1 + \frac{\beta_1 (\beta_1 - 1)}{2} \bar{C}_x^2 - \beta_1 \bar{C}_{yx} \right\} + 3 \kappa_2 \left\{ 1 + \frac{3 \tau_2^2}{8} \bar{C}_x^2 - \frac{\tau_2}{2} \bar{C}_{yx} \right\} - 1 \right]$$  \hspace{1cm} (5.38)

and

$$AV(T_3) = \bar{y}^2 \left[ 3 \kappa_1^2 N_{31} + 3 \kappa_2^2 N_{32} - 2 \kappa_1 \kappa_2 N_{33} - 2 \kappa_1 N_{34} + 2 \kappa_1 \kappa_2 N_{35} \right]$$  \hspace{1cm} (5.39)

Minimisation of $AV(T_3)$ is achieved for the optimum choice of constants $3 \kappa_1$ and $3 \kappa_2$

$$3 \kappa_1^* = \left( \frac{N_{32} N_{33} - N_{34} N_{35}}{N_{31} N_{32} - N_{35}^2} \right) \quad \text{and} \quad 3 \kappa_2^* = \left( \frac{N_{31} N_{34} - N_{33} N_{35}}{N_{31} N_{32} - N_{35}^2} \right)$$

where

$$N_{31} = \left[ 1 + \bar{C}_x^2 + \beta_1 \bar{C}_{yx} - 4 \beta_1 \bar{C}_{yx} \right].$$
\(N_{32} = \left[ 1 + \tilde{C}_y^2 + \tau_3^2 \tilde{C}_x^2 - 2 \tau_3 \tilde{C}_y \right],\)

\(N_{33} = \left[ 1 - \frac{\beta_1 (\beta_1 - 1)}{2} \tilde{C}_x^2 - \beta_1 \tilde{C}_y \right],\)

\(N_{34} = \left[ 1 + \frac{3 \tau_3^2}{8} \tilde{C}_x^2 - \frac{\tau_3}{2} \tilde{C}_y \right]\)

\(N_{25} = \left[ 1 + \tilde{C}_y^2 + \phi_6 \tilde{C}_x^2 - 2 \phi_7 \tilde{C}_y \right]\)

Such that, \(\tau_3 = \frac{\eta_1 \bar{X}}{\eta_1 \bar{X} + \lambda_1},\) \(\phi_6 = \frac{3 \tau_1^2}{8} + \frac{\beta_1 \tau_3}{2} - \frac{\beta_1 (\beta_1 - 1)}{2}\) and \(\phi_7 = 2 \beta_1 + \tau_3\)

Now assuming that \(\bar{X}\) is unknown, then the analogue of \(T_3\) becomes

\[T'_3 = \bar{y}_{CSS} \left[ 3 \kappa_{d1} \left( 2 - \frac{\bar{x}_{CSS}}{\bar{x}_{CSS}} \right) \right]^h + 3 \kappa_{d2} \exp \left( \frac{\eta_1 \left( \bar{x}_{CSS} - \bar{x}_{CSS} \right)}{\bar{x}_{CSS} + \bar{x}_{CSS} + 2 \lambda_1} \right) \]

(5.40)

where \(\left( 3 \kappa_{d1}, 3 \kappa_{d2}, \beta_1, \eta_1, \lambda_1 \right)\) are already defined earlier.

Then the asymptotic bias and variance of \(T'_3\) can be the same like \(T_3\). Because the difference between \(T_3\) and \(T'_3\) will be the same like the difference which is in \(T_1\) and \(T'_1\) i.e. the presence of \(\left( \tilde{C}_x^2, \tilde{C}_y \right)\) instead of \(\left( \tilde{C}_x^2, \tilde{C}_y \right)\).

There are many ways to construct the classes of proposed estimators \(T_1, T_2\) and \(T_3\). Many authors have discussed different family of estimators using known parameters and it is observed that, it increases the efficiency of the estimators. Thus we have given some members of the classes \((T_1, T_2, T_3)\) using known population parameters.

<table>
<thead>
<tr>
<th>Estimators</th>
<th>Suitable chosen constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T_{11})</td>
<td>(\kappa_1 \bar{y}_C + \kappa_2 (\bar{x}_C - N) + (1 - \kappa_1) \kappa_2 (\bar{X} - N) \exp \left[ C_x \frac{\bar{x}_C - \bar{X}}{\bar{x}_C + \bar{X} - 2N} \right] )</td>
</tr>
<tr>
<td>(T_{12})</td>
<td>(\bar{y}_C + \kappa_2 (\bar{x}_C - S_x) + (1 - \kappa_1) \kappa_2 (\bar{X} - S_x) \exp \left[ C_x^2 \frac{\bar{x}_C - \bar{X}}{\bar{x}_C + \bar{X} - 2S_x} \right] )</td>
</tr>
<tr>
<td>(T_{13})</td>
<td>(-\kappa_1 \bar{y}_C + \kappa_2 (\bar{x}_C - S_x) - (1 - \kappa_1) \kappa_2 (\bar{X} - S_x) \exp \left[ \frac{\bar{x}_C - \bar{X}}{\bar{x}_C + \bar{X} - 2S_x} \right] )</td>
</tr>
<tr>
<td>(T_{14})</td>
<td>(\bar{y}_C + \kappa_2 (N \bar{x}_C - \rho_s) + (1 - \kappa_1) \kappa_2 (N \bar{X} - \rho_s) \exp \left[ \frac{N(\bar{x}_C - \bar{X})}{N(\bar{x}_C + \bar{X}) - 2 \rho_s} \right] )</td>
</tr>
</tbody>
</table>
Table 5.4 Some Members of Classes of Estimator $T_2$

<table>
<thead>
<tr>
<th>Estimators</th>
<th>Suitably chosen constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{21} = \bar{y}_c \left[ 2 \kappa_1 + 2 \kappa_2 \left{ 2 - \frac{x_c}{\bar{x}} \right} \right] \exp \left[ \frac{x_c - \bar{x}}{2 + x_c + \bar{x}} \right]$</td>
<td>$\mu$ $v$ $\beta$ $\gamma$</td>
</tr>
<tr>
<td>$T_{22} = \bar{y}_c \left[ 2 \kappa_1 + 2 \kappa_2 \left{ 2 - \frac{x_c}{\bar{x}} \right} \right] \exp \left[ \frac{\rho_x (x_c - \bar{x})}{2 + \rho_x (x_c + \bar{x})} \right]$</td>
<td>$\rho_x$ $1$ $1$ $-1$</td>
</tr>
<tr>
<td>$T_{23} = \bar{y}_c \left[ 2 \kappa_1 + 2 \kappa_2 \left{ 2 - \frac{x_c}{\bar{x}} \right} \right] \exp \left[ \frac{(x_c - \bar{x})}{(x_c + \bar{x}) - 2 \rho_x} \right]$</td>
<td>$-1$ $\rho_x$ $1$ $-1$</td>
</tr>
<tr>
<td>$T_{24} = \bar{y}_c \left[ 2 \kappa_1 + 2 \kappa_2 \left{ 2 - \frac{x_c}{\bar{x}} \right} \right] \exp \left[ \frac{(x_c - \bar{x})}{(x_c + \bar{x}) + 2N} \right]$</td>
<td>$1$ $N$ $1$ $-1$</td>
</tr>
</tbody>
</table>

Table 5.5 Some Members of Classes of Estimator $T_3$

<table>
<thead>
<tr>
<th>Estimators</th>
<th>Suitably chosen constants</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{31} = \bar{y}_{css} \left[ 3 \kappa_1 \left{ 2 - \frac{x_c}{\bar{x}} \right} + 3 \kappa_2 \exp \left[ \frac{(x - x_c)}{2 + (x + x_c)} \right] \right]$</td>
<td>$\eta_1$ $\lambda_1$ $\beta_1$</td>
</tr>
<tr>
<td>$T_{32} = \bar{y}_{css} \left[ 3 \kappa_1 \left{ 2 - \frac{x_c}{\bar{x}} \right} + 3 \kappa_2 \exp \left[ \frac{(x - x_c)}{(x + x_c) + 2N} \right] \right]$</td>
<td>$-1$ $-N$ $1$</td>
</tr>
<tr>
<td>$T_{33} = \bar{y}_{css} \left[ 3 \kappa_1 \left{ 2 - \frac{x_c}{\bar{x}} \right} + 3 \kappa_2 \exp \left[ \frac{(x - x_c)}{(x + x_c) - 2S_x} \right] \right]$</td>
<td>$1$ $-S_x$ $1$</td>
</tr>
<tr>
<td>$T_{34} = \bar{y}_{css} \left[ 3 \kappa_1 \left{ 2 - \frac{x_c}{\bar{x}} \right} + 3 \kappa_2 \exp \left[ \frac{(x - x_c)}{(x + x_c) + 2S_x} \right] \right]$</td>
<td>$1$ $S_x$ $1$</td>
</tr>
</tbody>
</table>

Note: Here lowercase ‘C’ stands for CSS (circular systematic sampling)
5.2.4 Efficiency Comparison

In order to compare the performance of the proposed classes of estimators based on CSS with the estimators based on SRSWOR, we can use the following estimators \((\bar{y}, \bar{y}_h, \bar{y}_h^{'})\) in SRSWOR

\[
V(\bar{y}) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2
\]

\[
AV(\bar{y}_h) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2[1 - \rho_{yx}^2]
\]

and

\[
AV(\bar{y}_h^{'}) = \left(\frac{1}{n} - \frac{1}{N}\right)S_y^2 + \left(\frac{1}{n'} - \frac{1}{N'}\right)S_y^2[1 - \rho_{yx}^2]
\]

Note: Here using SRSWOR design, at first phase a large sample \(s'\) of size \(n'\) (\(n' < N\)) is selected randomly to estimate \(\bar{X}\) only. In second phase, a sub-sample \(s\) of size \(n\) form \(n'\) units is drawn randomly to estimate \(\bar{Y}\) where \(n' = m n\).

It is not easy to make analytical comparison of the proposed classes of estimators. We compute variance and minimum AV of the considered estimators in CSS along with the variance and minimum AV of the estimators in SRSWOR. For this purpose, we use population data set earlier considered by Koyuncu and Kadilar (2009) and Singh and Solanki (2013). The data concerns primary and secondary schools of 923 districts of Turkey in 2007. The description of variables is given below

\(y\) = number of teachers in both primary and secondary school;

\(x\) = number of students in both primary and secondary school.

\(N=923\quad n'=360\quad n=180\quad m=2\quad \bar{X}=11440.5\quad \bar{Y}=436.43\)

\(S_y=749.94\quad S_x=21331.13\quad \rho_{yx}=0.9543\quad \rho_y=-0.00255\quad \rho_x=-0.00316\)

For two-phase, one can select \(1 < m < 5\) (as we mentioned earlier \(1 < m < k\)). All possible values of \(m\) are considered and numerical results are provided only for \(m = 2\). Because it is observed that for \(m = 2\), all the considered estimators are more efficient in CSS than SRSWOR. For \(m = 3\) and \(m= 4\), the estimators under SRSWOR perform a little better than CSS. So in this numerical example \(m = 2\) can be the best choice among others.

Following is the description of the considered estimators in Table 5.6 The estimators \((\bar{y}, \bar{y}_h, \bar{t}_*)\) are based on SRSWOR and \((\bar{y}_{CSS}, \bar{y}_h^{'}, T_*\) on CSS, further these estimators are also considered for two phase sampling.
Table 5.6: Minimum Asymptotic variance of considered estimators under single and two phase sampling-

<table>
<thead>
<tr>
<th>Estimators</th>
<th>V/AV under single phase</th>
<th>V/ AV under Two phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SRSWOR</td>
<td>CSS</td>
</tr>
<tr>
<td>$\bar{y}$</td>
<td>2515.17</td>
<td>1698.75</td>
</tr>
<tr>
<td>$\bar{y}_{lr}$</td>
<td>224.62</td>
<td>151.71</td>
</tr>
<tr>
<td>$T_{11}$</td>
<td>169.71</td>
<td>135.90</td>
</tr>
<tr>
<td>$T_{12}$</td>
<td>188.12</td>
<td>141.38</td>
</tr>
<tr>
<td>$T_{13}$</td>
<td>153.95</td>
<td>131.49</td>
</tr>
<tr>
<td>$T_{14}$</td>
<td>218.17</td>
<td>149.58</td>
</tr>
<tr>
<td>$T_{21}$</td>
<td>110.46</td>
<td>108.96</td>
</tr>
<tr>
<td>$T_{22}$</td>
<td>108.01</td>
<td>107.15</td>
</tr>
<tr>
<td>$T_{23}$</td>
<td>110.44</td>
<td>108.96</td>
</tr>
<tr>
<td>$T_{24}$</td>
<td>123.63</td>
<td>113.37</td>
</tr>
<tr>
<td>$T_{31}$</td>
<td>206.54</td>
<td>141.18</td>
</tr>
<tr>
<td>$T_{32}$</td>
<td>205.57</td>
<td>141.17</td>
</tr>
<tr>
<td>$T_{33}$</td>
<td>203.69</td>
<td>141.13</td>
</tr>
<tr>
<td>$T_{34}$</td>
<td>201.82</td>
<td>141.08</td>
</tr>
</tbody>
</table>

In Table 5.6, it can be seen that the variance of $\bar{y}_{CSS}$ is smaller than the variance of $\bar{y}$. Also, the asymptotic variance of considered estimator under CSS is smaller than under SRSWOR. Hence, we conclude that the estimators based on CSS are more efficient than the estimator based on SRSWOR. Note that $\rho_y$ and $\rho_x$ both are less than $-\frac{1}{(N-1)}$. It is also observed that all the considered estimators $T_{i(1)}, T_{2(1)}, T_{3(1)}$ are more efficient than the regression estimator $\bar{y}_{lr}$. Furthermore, in first phase sampling the estimator $T_{13}$ in class $T_1$, $T_{22}$ in class $T_2$, and $T_{34}$ in class $T_3$, provides minimum asymptotic variance among others but in two phase sampling $T_{11}$ in class $T_1$, $T_{23}$ in class $T_2$, and $T_{31}$ in class $T_3$, provides efficient result. Henceforth, the estimator $T_{22}$ (in 1st phase) and $T_{11}$ (in two phase), results the best one in terms of efficiency among all considered estimators. Hence, from Tables 3, we can conclude that the class $T_2$ in case of single-phase and $T_1$ for two-phase may be the best choice among others.
5.2.5 Non-Response Problem under CSS

When a sample of size \( n \) is selected from \( N \) circular systematic samples to collect information of \( Y \), then incomplete or missing information might be present. The reasons non-response problem occurrence may vary in different situations. For instance, the reasons of non-response in the data set considered in previous the section may be due to strikes, holidays etc. When non-response occurs in a CSS, we can follow the well-known Hansen and Hurwitz (1946) non-respondents sub-sampling technique. Suppose that \( n_1 \) units out of \( n \) can supply information on \( Y \) and remaining \( n_2 = n - n_1 \) units are taken as non-respondents. Following the technique of Hansen and Hurwitz (1946), a sub sample of size \( n_r = \frac{n_2}{l}, (l > 1) \) is selected by SRSWOR from \( n_2 \) non-respondent units. Assume that all \( n_r \) units show full response on Second call (of course \( nr \) must be an integer and if it isn’t so, it is necessary to round). The population is said to be divided into two groups \( U_1 \) and \( U_2 \) of sizes \( N_1 \) and \( N_2 \), where \( U_1 \) is a group of respondents that would give response on the first call and \( U_2 \) is non-respondents group which could respond on the second call. Obviously \( N_1 \) and \( N_2 \) are unknown quantities.

The estimator proposed by Hansen and Hurwitz (1946) is given by-

\[
\bar{y}^0 = d_1 \bar{y}_1 + d_2 \bar{y}_2
\]

where

\[
\bar{y}_1 = \frac{\sum_{j=1}^{n_1} y_{ij}}{n_1}, \quad \bar{y}_2 = \frac{\sum_{j=1}^{n_2} y_{ij}}{n}, \quad d_1 = \frac{n_1}{n} \quad \text{and} \quad d_2 = \frac{n_2}{n}
\]

The estimator \( \bar{y}^* \) is unbiased and has variance

\[
V(\bar{y}^0) = \tilde{S}_y^2 + wS_{y(2)}^2 = \tilde{S}_y^2
\]

where

\[
\tilde{S}_y^2 = \frac{\tilde{S}_y^2}{Y_2^2}, \quad w = \frac{N_2 (l - 1)}{n N}, \quad Y_2 = \frac{\sum_i y_{ij}}{N_2} \quad \text{and} \quad S_{y(2)}^2 = \frac{\sum_i \sum_j (y_{ij} - \bar{Y}_2)^2}{n_2 (N_2 - 1)}
\]

The linear regression estimator defined in equation (5.20), in case of non-response in \( Y \), can be written as
\[ \bar{y}_\ell (c) = \bar{y}_{\text{CSS}} + \hat{\beta}_{xy} (X - \bar{x}_{\text{CSS}}) \] (5.43)

The asymptotic variance of \( \bar{y}_\ell (c) \) is given by

\[ \text{AV} (\bar{y}_\ell (c)) = \tilde{S}_y^2 (1 - \tilde{\rho}_{xy}^2) + wS_{y(2)}^2 \] (5.44)

When there is a non-response in Y and \( \bar{X} \) is unknown, then (5.22) becomes

\[ \bar{y}_\ell (c) = \bar{y}_{\text{CSS}} + \hat{\beta}_{xy} (\bar{x}_{\text{CSS}} - \bar{x}_{\text{CSS}}) \] (5.45)

The asymptotic variance of \( \bar{y}_\ell (c) \) is given by

\[ \text{AV} (\bar{y}_\ell (c)) = \tilde{S}_y^2 \left( 1 - \tilde{\rho}_{xy}^2 \right) + wS_{y(2)}^2 \] (5.46)

where, \( \tilde{\rho}_{xy}^2 = \frac{\tilde{S}_{xy}^2}{\tilde{S}_y^2 \tilde{S}_x^2} \) are such that

### 5.2.6 Suggested Estimators under Non-Response

Now the suggested classes \((T_1, T_2, T_3)\) in presence of non-response in Y, could be expressed as

\[ T_1^O = \left[ \kappa_1 \hat{y}_{\text{CSS}} + \kappa_2 \bar{x}_{\text{CSS}} + \left( 1 - \kappa_1 - \kappa_2 \right) \bar{X} \right] \exp \left[ \frac{\alpha (\bar{X} - \bar{x}_{\text{CSS}})}{\bar{X} + \bar{x}_{\text{CSS}}} \right] \] (5.47)

\[ T_2^O = \bar{y}_{\text{CSS}} \left[ \kappa_1 + \kappa_2 \left( 2 - \bar{x}_{\text{CSS}} \right) \right] \left( \frac{\gamma (\mu + v)}{(\mu + v) + (\mu \bar{x}_{\text{CSS}} + v)} \right) \] (5.48)

and

\[ T_3^O = \bar{y}_{\text{CSS}} \left[ 3 \kappa_1 \left( 2 - \bar{x}_{\text{CSS}} \right) \right]^{\kappa_1} \left[ 3 \kappa_2 \exp \left( \eta_1 \left( \bar{X} - \bar{x}_{\text{CSS}} \right) \right) \right] \left( \frac{2\lambda_1}{\bar{X} + \bar{x}_{\text{CSS}}} \right) \] (5.49)

The asymptotic bias of \((T_1^O, T_2^O, T_3^O)\) will be same of \((T_1, T_2, T_3)\). The minimum asymptotic variance of \((T_1^O, T_2^O, T_3^O)\) under non response is respectively given by

\[ \text{AV} (T_1^O) = \left[ N + \kappa_1 \kappa_2 N_{11} + \kappa_2^2 N_{12} + 2 \kappa_2 \kappa_1 N_{i1} + 2 \kappa_2 N_{14} + 2 \kappa_1 \kappa_2 N_{15} \right] \] (5.50)

Minimisation of \( \text{AV} (T_1^O) \) is achieved for the optimum choice of constants \( \kappa_1 \) and \( \kappa_2 \)

\[ \kappa_1^* = \left( \frac{N_{12} N_{13} - N_{14} N_{15}}{N_{11} N_{12} - N_{15}^2} \right) \text{ and } \kappa_2^* = \left( \frac{N_{12} N_{14} - N_{13} N_{15}}{N_{11} N_{12} - N_{15}^2} \right) \]

where
\[ N = \phi^2 + \frac{\alpha^2 \tau_1^2 \ddot{X}^2 \ddot{C}_x^2}{2} + 2\phi \phi_1 \dddot{C}_x \dddot{x}, \quad N_{11} = Y^2 \dddot{C}_y^2 + \phi^2 + \frac{\alpha^2 \tau_1^2 \phi^2}{4} \dddot{C}_x^2 - 2\phi \phi_2 \dddot{C}_x^2 + 2\alpha \tau_1 \phi \dddot{Y} \dddot{C}_x \dddot{y} \]

\[ N_{12} = \eta^2 \dddot{X}^2 \dddot{C}_x^2, \quad N_{13} = \phi^2 + \frac{\alpha \tau_1 Y(\ddot{X} + \dddot{X})}{2} \dddot{C}_x \dddot{y} + \frac{\alpha^2 \tau_1^2 \phi^2 \dddot{C}_x^2}{2} + \phi \dddot{C}_x (\phi_1 - \phi_2) \]

\[ N_{14} = \eta \alpha \tau_1 X(\ddot{X} + \dddot{X}) \dddot{C}_x^2 / 2, \quad N_{15} = \eta \alpha \tau_1 \phi \dddot{X} \dddot{C}_x^2 + \eta \dddot{Y} \dddot{C}_x \dddot{y} \]

\[
\text{AV} \left( T_2^0 \right) = Y^2 \left[ \kappa^2 \frac{N_{21}}{N_{21} N_{22} - N_{25}^2} N_{24} - 2^2 \kappa_1 N_{23} - 2^2 \kappa_2 N_{24} + 2^2 \kappa_1^2 \kappa_2 N_{25} \right]
\]

Minimisation of \( \text{AV} \left( T_2^0 \right) \) is achieved for the optimum choice of constants \( \kappa^2 \) and \( \kappa_2^2 \)

\[
2 \kappa_1^* = \left( \frac{N_{22} N_{23} - N_{24} N_{25}}{N_{21} N_{22} - N_{25}^2} \right) \quad \text{and} \quad 2 \kappa_2^* = \left( \frac{N_{21} N_{24} - N_{23} N_{25}}{N_{21} N_{22} - N_{25}^2} \right)
\]

where

\[ N_{21} = \left[ 1 + \dddot{C}_y^2 + \frac{\gamma^2 \tau_2^2}{4} + 2\phi_3 \right] \dddot{C}_x^2 - 2\gamma \tau_2 \dddot{C}_x \dddot{y}, \]

\[ N_{22} = \left[ 1 + \dddot{C}_y^2 + \{\phi_3 + 2\phi_2\} \dddot{C}_x^2 - 4\phi_3 \dddot{C}_x \dddot{y} \right], \]

\[ N_{23} = \left[ 1 + \phi_3 \dddot{C}_x^2 - \frac{\gamma \tau_2 \dddot{C}_x \dddot{y}}{2} \right], \quad N_{24} = \left[ 1 + \phi_4 \dddot{C}_x^2 - \phi_3 \dddot{C}_x \dddot{y} \right] \]

\[ N_{25} = \left[ 1 + \dddot{C}_y^2 + \phi_4 \dddot{C}_x^2 - 2\phi_5 \dddot{C}_x \dddot{y} - \beta \tau_2 \dddot{C}_x \dddot{y} + \frac{\gamma \tau_2 \phi_5 \dddot{C}_x^2}{2} + \phi_3 \dddot{C}_x \dddot{y} \right] \]

\[
\text{AV} \left( T_3^0 \right) = Y^2 \left[ \kappa_1 \frac{N_{31}}{N_{31} N_{32} - N_{35}^2} N_{33} + \kappa_2 \frac{N_{32}}{N_{31} N_{32} - N_{35}^2} N_{34} - 2 \kappa_1 N_{33} - 2 \kappa_2 N_{34} + 2 \kappa_1 \kappa_2 N_{35} \right]
\]

Minimisation of \( \text{AV} \left( T_3^0 \right) \) is achieved for the optimum choice of constants \( \kappa_1^3 \) and \( \kappa_2^3 \)

\[
3 \kappa_1^* = \left( \frac{N_{32} N_{33} - N_{34} N_{35}}{N_{31} N_{32} - N_{35}^2} \right) \quad \text{and} \quad 3 \kappa_2^* = \left( \frac{N_{31} N_{34} - N_{33} N_{35}}{N_{31} N_{32} - N_{35}^2} \right)
\]

where

\[ N_{31} = \left[ 1 + \dddot{C}_y^2 + \beta_1 \dddot{C}_x^2 - 4\beta_1 \dddot{C}_x \dddot{y} \right], \quad N_{32} = \left[ 1 + \dddot{C}_y^2 + \tau_3^2 \dddot{C}_x^2 - 2\tau_3 \dddot{C}_x \dddot{y} \right]. \]

\[ N_{33} = \left[ 1 - \frac{\beta_1 (\beta_1 - 1)}{2} \dddot{C}_x^2 - \beta_1 \dddot{C}_x \dddot{y} \right], \quad N_{34} = \left[ 1 + \frac{3\tau_2^2}{8} \dddot{C}_x^2 - \frac{\tau_3^2}{2} \dddot{C}_x \dddot{y} \right] \]

\[ N_{25} = \left[ 1 + \dddot{C}_y^2 + \phi_6 \dddot{C}_x^2 - 2\phi_3 \dddot{C}_x \dddot{y} \right] \]

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When $\bar{X}$ is unknown, the biases of the classes $\left( T_1^0, T_2^0, T_3^0 \right)$ will be same of $(T_1', T_2', T_3')$. For the minimum asymptotic variances of these classes, only replacing the terms $\left( \tilde{C}_x^2, \tilde{C}_{yx} \right)$ by and $\left( \tilde{C}_x^2, \tilde{C}_{yx} \right)$.

### 5.2.7 Efficiency Comparison

To compare the performance of the suggested estimators based on CSS with the estimators based on SRSWOR in the presence of non-response in $Y$, we can use the following estimators $(\overline{y}^0, \overline{y}_{lr}^0, \overline{y}_{lr}^{0(0)})$ in SRSWOR

$$
V(\overline{y}^0) = \left( \frac{1}{n} - \frac{1}{N} \right) S_y^2 + wS_{y(2)}^2
$$

$$
AV(\overline{y}_{lr}^0) = \left( \frac{1}{n} - \frac{1}{N} \right) S_y^2 [1 - \rho_{yx}^2] + wS_{y(2)}^2
$$

And

$$
AV(\overline{y}_{lr}^{0(0)}) = \left( \frac{1}{n'} - \frac{1}{N} \right) S_y^2 \left( \frac{1}{n} - \frac{1}{n'} \right) S_{y(2)}^2 [1 - \rho_{yx}^2] + wS_y^2
$$

We take all the possibilities for weights of the missing values (10%, 20%, 30%, 40%) etc and observe that the relative efficiency of the considered estimators is not affected by different weights of missing values. Although numerical results are different for different weights, the behavior of results is similar in all cases. Hence, numerical results are provided only for 10% weight of missing values and consider last 92 values as non-respondents.

$$
\overline{Y}_2 = 522.80 \quad S_{y(2)} = 876.42 \quad N_2 = 92 \quad l = 2
$$

Due to presence of non-response, extra variability is introduced in estimators. As expected, the variability of all considered estimators with incomplete information (see Tables 5.7) is higher than the estimators with complete response (Tables 5.6). Moreover, as expected, for $l > 2$, the asymptotic variance of the estimators become higher, that’s why we are taking only the case when $l=2$.

Following is the description of the considered estimators in Table 5.7. The estimators $(\overline{y}, \overline{y}_{lr}, \bar{t}^0)$ are based on SRSWOR and $(\overline{y}_{CSS}, \overline{y}_{lr}, \bar{t}^0)$ on CSS, further these estimators are also considered for two phase sampling.
Table 5.7: The minimum V/AV of the considered estimators

<table>
<thead>
<tr>
<th>Estimators</th>
<th>V/AV under single phase</th>
<th>V/AV under Two phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SRSWOR</td>
<td>CSS</td>
</tr>
<tr>
<td>(\bar{y}^O)</td>
<td>2940.51</td>
<td>2124.09</td>
</tr>
<tr>
<td>(\bar{y}_{lr}^O)</td>
<td>649.96</td>
<td>577.05</td>
</tr>
<tr>
<td>(T_{11}^O)</td>
<td>595.385</td>
<td>562.17</td>
</tr>
<tr>
<td>(T_{12}^O)</td>
<td>614.900</td>
<td>568.25</td>
</tr>
<tr>
<td>(T_{13}^O)</td>
<td>639.042</td>
<td>574.60</td>
</tr>
<tr>
<td>(T_{14}^O)</td>
<td>644.75</td>
<td>576.33</td>
</tr>
<tr>
<td>(T_{21}^O)</td>
<td>558.138</td>
<td>547.83</td>
</tr>
<tr>
<td>(T_{22}^O)</td>
<td>553.233</td>
<td>546.31</td>
</tr>
<tr>
<td>(T_{23}^O)</td>
<td>558.123</td>
<td>547.83</td>
</tr>
<tr>
<td>(T_{24}^O)</td>
<td>569.94</td>
<td>551.53</td>
</tr>
<tr>
<td>(T_{31}^O)</td>
<td>640.865</td>
<td>573.34</td>
</tr>
<tr>
<td>(T_{32}^O)</td>
<td>640.098</td>
<td>573.32</td>
</tr>
<tr>
<td>(T_{33}^O)</td>
<td>638.609</td>
<td>573.28</td>
</tr>
<tr>
<td>(T_{34}^O)</td>
<td>637.1094</td>
<td>573.24</td>
</tr>
</tbody>
</table>

From table 5.7, we can see that \(T_{11}^O, T_{12}^O, T_{13}^O\) are more efficient than the regression estimator \(\bar{y}_{lr}^O\). Furthermore, among all the classes of estimators \(T_{22}^O\) is most efficient (both in single and two phase sampling).

5.2.8 Conclusion

In the present study some improved estimators are proposed and their properties are studied under circular systematic sampling. The problem is also extended to the case of non-response in CSS. From table 5.6 and 5.7, it has been observed that the proposed estimators are more efficient in CSS than SRSWOR for single-phase and two-phase sampling.
Appendix

In this section, we have given the complete procedure for obtaining the asymptotic bias and asymptotic variance of the suggested estimators \((T_1, T_2, T_3)\). From equation (5.25), \(T_1\) in terms of \(\delta_i\)'s, could be expressed as

\[
T_1 = \left[1, \kappa, Y(1+\delta_{y})^{-1}\kappa_2\{\eta(1+\delta_{x})X+\lambda\}+(1^{-1}\kappa_1^{-1}\kappa_2)\right] \exp \left[\alpha \frac{\bar{X}' - \{\eta(1+\delta_{x})\bar{X} + \lambda\}}{\bar{X}' + \{\eta(1+\delta_{x})\bar{X} + \lambda\}}\right]
\]

On retaining only the terms up to the second degree of \(\delta_i\)'s, we have

\[
T_1 - \bar{Y} = \left[\phi - \frac{\alpha \tau_1 \bar{X}'}{2} - \delta_{x} + \phi_1 \delta_{x}^2 + \kappa_1 \left(\delta_{y} \bar{Y} - \phi + \frac{\alpha \tau_1 \phi}{2} \delta_{x} + \phi_2 \delta_{x}^2 - \frac{\bar{Y} \alpha \tau_1 \delta_{y} \delta_{x}}{2}\right)\right]
\]

\[
+ \kappa_2 \left\{\eta \bar{X} \delta_{x} - \frac{\eta \bar{X} \alpha \tau_1 \delta_{y}^2}{2}\right\}
\]

(A.5.1)

After taking expectation both sides of equation (34), we get the asymptotic bias of \(T_1\) as

\[
AB(T_1) = \left[\phi + \phi_1 \bar{C}_x^2 + \kappa_1 \left\{\phi_2 \bar{C}_x^2 - A - \frac{\bar{Y} \alpha \tau_1 \bar{C}_{yx}}{2}\right\} - \kappa_2 \frac{\eta \bar{X} \alpha \tau_1 \bar{C}_x^2}{2}\right]
\]

Squaring and taking expectations of both sides of equation (A.5.2), we have the required asymptotic variance of \(T_1\), respectively given by equation (5.28).

Now from equation (5.33), \(T_2\) in terms of \(\delta_i\)'s, could be expressed as

\[
T_2 = (1+\delta_{y}) \bar{Y} \left[2 \kappa_1 + \kappa_2 \left\{2 - \frac{(1+\delta_{x})\bar{X}}{X}\right\}\right] \exp \left[\gamma \frac{(\mu \bar{X} + \nu) - (\mu(1+\delta_{x})\bar{X} + \nu)}{(\mu \bar{X} + \nu) + (\mu(1+\delta_{x})\bar{X} + \nu)}\right]
\]

(A.5.3)

After expanding equation (A.5.3), we have

\[
T_2 = \bar{Y} \left[2 \kappa_1 \left\{1 + \delta_{y} - \frac{\gamma \tau_2 \delta_{x}}{2} + \phi_3 \delta_{x}^2 - \frac{\gamma \tau_2 \delta_{x} \delta_{y}}{2}\right\}\right] + \kappa_2 \left\{1 + \delta_{y} - \phi_3 \delta_{x} + \phi_4 \delta_{x}^2 - \phi_5 \delta_{x} \delta_{y}\right\}
\]

(A.5.4)

Subtracting \(\bar{Y}\) and taking expectations both sides of above equation (A.5.4), we get the requires bias of \(T_2\) as

\[
AB(T_2) = \bar{Y} \left[2 \kappa_1 \left\{1 + \phi_3 \bar{C}_x^2 - \frac{\gamma \tau_2 \bar{C}_{yx}}{2}\right\}\right] + \kappa_2 \left\{1 + \phi_4 \bar{C}_x^2 - \phi_5 \bar{C}_{yx}\right\} - 1
\]

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Subtracting and squaring and then taking expectations of both sides of equation (A.5.4), we have the required asymptotic variance of \( T_2 \), respectively given by equation (5.35).

Now from equation (5.37), \( T_3 \) in terms of \( \delta_1, \delta_2, \delta_3 \), could be expressed as

\[
T_3 = (1 + \delta_x) \left[ 3k_1 \{2 - (1 - \delta_x)^\beta_1\} + 3k_2 \exp \left\{ -\frac{\eta_1 \bar{X}\delta_x}{2(\eta_1 \bar{X} + \lambda_1) + \eta_1 \bar{X}\delta_x} \right\} \right]
\]  
(A.5.5)

After expanding equation (A.5.5), we have

\[
T_3 = \bar{Y} \left[ 3k_1 \left\{1 + \delta_x - \beta_1 \delta_x - \frac{\beta_1 (\beta_1 - 1)}{2} \delta_x^2 - \beta_1 \delta_x \delta_y \right\} + 3k_2 \left\{1 + \delta_x - \frac{\tau_3 \delta_x}{2} + \frac{3\tau_3^2 \delta_x^2}{8} - \frac{\tau_3}{2} \delta_x \delta_y \right\} \right]
\]  
(A.5.6)

Subtracting \( \bar{Y} \) and taking expectations both sides of above equation (A.5.6), we get the requires bias of \( T_3 \) as

\[
\text{AB}(T_3) = \bar{Y} \left[ 3k_1 \left\{1 + \beta_1 (\beta_1 - 1) \bar{C}_x^2 - \beta_1 \bar{C}_{xy} \right\} + 3k_2 \left\{1 + \frac{3\tau_3^2 \bar{C}_x^2}{8} - \frac{\tau_3}{2} \bar{C}_{xy} \right\} - 1 \right]
\]

Subtracting and squaring and then taking expectations of both sides of equation (A.5.6), we have the required asymptotic variance of \( T_3 \), respectively given by equation (5.39).

Note: The optimum values of \( i \kappa_1 \) and \( i \kappa_2 \) can be obtained by differentiating their respective
AV(T) partially w.r.t. to \( i \kappa_1 \) and \( i \kappa_2 \) and equate it to zero as

\[
\frac{\partial \text{AV}(T)}{\partial i \kappa_1} = 0 \quad \text{and} \quad \frac{\partial \text{AV}(T)}{\partial i \kappa_2} = 0
\]  
(A.5.7)

After solving (A.5.7) for \( i \kappa_1 \) and \( i \kappa_2 \), we get the required optimum values for \( i \kappa_1 \) and \( i \kappa_2 \).